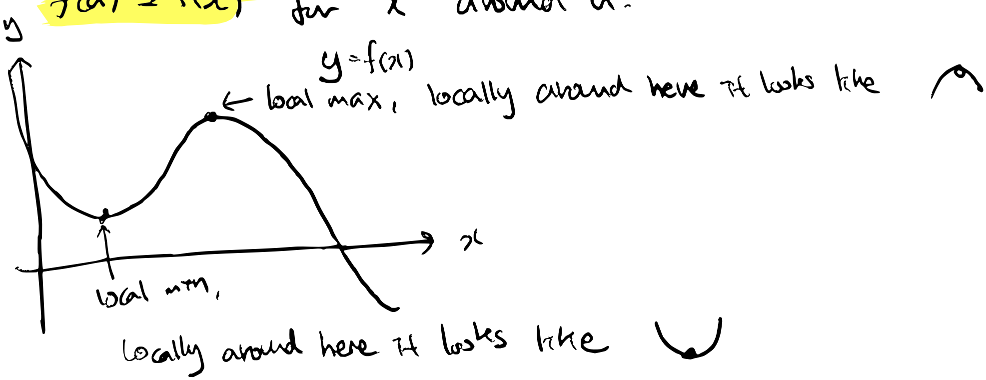
Local minima / maxima

There is the notion of local minimum / maximum in several variables case, just as in the case of one-variable.

One-variable reminder

$f(x)$ has a local minimum at $x=a$ if $f(a) \le f(x)$ for x around a.

$f(x)$ has a local maximum at $x=a$ if $f(a) \ge f(x)$ for x around a.



Theorem Local min/max are either critical points ($f'(x)=0$) or boundary of the region.

Boundary of the domain case looks like 

local max, around here it looks like

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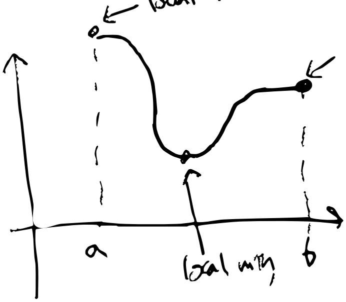
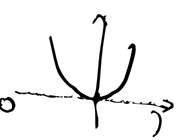
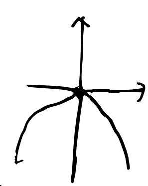
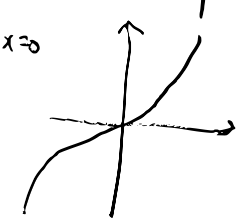


around here it looks like 

How can you tell if a critical point is a local minimum/maximum?

Critical points have the second derivative test.

<u>Second derivative test</u> If a is a critical point of $f(x)$ ($f'(a)=0$), then

* $f''(a) > 0 \Rightarrow$ local minimum ($y=x^2$ at $x=0$ )
* $f''(a) < 0 \Rightarrow$ local maximum ($y=-x^2$ at $x=0$ )
* $f''(a) = 0 \Rightarrow$ not decided. ($y=x^3$ at $x=0$ )

<u>Example</u> Find the critical points of $f(x) = x^3 - 3x$

and determine whether they are local min/max.

<u>Solution</u> $f'(x) = 3x^2 - 3$, so $f'(x) = 0$ means $3x^2 = 3$, or

$x^2 = 1$, or $x = 1$ or -1.

$f''(x) = 6x$, so $f''(1) > 0 \Rightarrow 1$ is local min

$f''(-1) < 0 \Rightarrow -1$ is local max.

In the two-variable case there are analogous definitions.

<u>Def</u> A point (a,b) is a critical point of $f(x,y)$ if

$f_x(a,b) = 0$ and $f_y(a,b) = 0$. Equivalently, (a,b) is a

critical point if $\nabla f(a,b) = \vec{0}$.

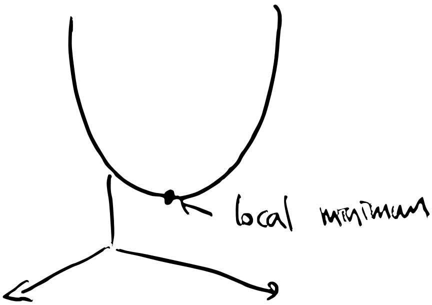
<u>Def</u> A point (a,b) is a local minimum of $f(x,y)$ if

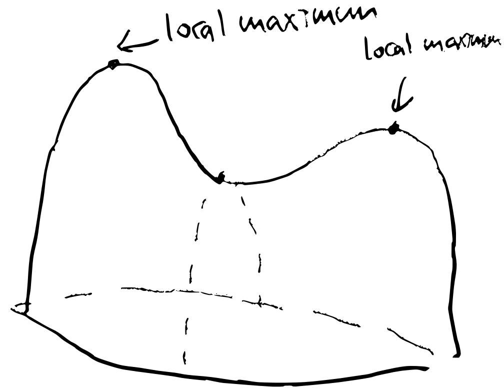
$f(x,y) \geq f(a,b)$ for all (x,y) around (a,b).

A point (a,b) is a local maximum of $f(x,y)$ if $f(x,y) \leq f(a,b)$

for all (x,y) around (a,b).

Theorem Local min/max are either critical points or on the boundary of the region





Critical points have the second derivative test.

Second derivative test (2-variable)

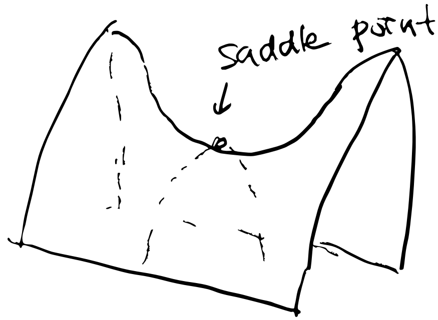
Let (a,b) be a critical point of $f(x,y)$.

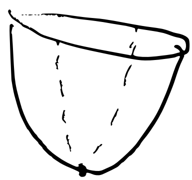
Namely $f_x(a,b) = 0$, $f_y(a,b) = 0$.

Let $D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}(a,b)^2$

* $D(a,b) > 0$, $f_{xx}(a,b) > 0 \Rightarrow$ local minimum
* $D(a,b) > 0$, $f_{xx}(a,b) < 0 \Rightarrow$ local maximum
* $D(a,b) < 0 \Rightarrow$ saddle point
* Otherwise $\Rightarrow$ not decided.

Def A saddle point is something that looks like a saddle:



Local min:  Local max: 

A saddle point is a peak in one direction, a valley in another direction

Example $f(x,y) = x^3 + 3xy + y^3$

Find the critical points and check if they are local min/max.

Sol $f_x = 3x^2 + 3y$, $f_y = 3x + 3y^2$ $\Rightarrow$ $f_x(x,y) = f_y(x,y) = 0$ means

$-x^2 = y, \quad -y^2 = x$

$\Rightarrow -x^4 = x$, so either $x = 0$ or $x = -1$

$\Rightarrow y = 0, \quad y = -1$

The critical points are $(0,0)$, $(-1,-1)$.

We have

$f_{xx}(x,y) = 6x$

$f_{yy}(x,y) = 6y$

$f_{xy}(x,y) = 3$

$\Rightarrow D(x,y) = 36xy - 9$

For the critical point $(0,0) \Rightarrow D(0,0) = -9 < 0 \Rightarrow$ saddle point

For the critical point $(-1,-1) \Rightarrow D(-1,-1) = 36 - 9 > 0$

$f_{xx}(-1,-1) = -6 < 0$

$\Rightarrow$ local max.

Example $f(x,y) = x^2 + y^4 + 2xy$

Find the critical points and check if they are local min/max

$$\left.\begin{array}{l} f_x(x,y) = 2x + 2y \\ f_y(x,y) = 4y^3 + 2x \end{array}\right) \Rightarrow \begin{array}{l} f_x(x,y) = 0 \text{ means } y = -x \\ f_y(x,y) = 0 \text{ means } 2y^3 = -x = y \end{array}$$

So either $y = 0$ or $\frac{2y^3}{y} = \frac{y}{y} \Rightarrow 2y^2 = 1, \; y = \pm\sqrt{\frac{1}{2}}$

So critical points are $(0,0), \left(\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right), \left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$

$$\left.\begin{array}{l} f_{xx}(x,y)=2 \\ f_{xy}(x,y)=2 \\ f_{yy}(x,y)=12y^2 \end{array}\right\} \Rightarrow D(x,y)=24y^2-4$$

For the critical point $(0,0)$, $D(0,0)=-4<0 \Rightarrow$ saddle point.

For the critical point $\left(-\sqrt{\frac{1}{2}},\sqrt{\frac{1}{2}}\right)$, $D\left(-\sqrt{\frac{1}{2}},\sqrt{\frac{1}{2}}\right)=12-4>0 \Rightarrow$ local min

$$f_{xx}=2$$

For the critical point $\left(\sqrt{\frac{1}{2}},-\sqrt{\frac{1}{2}}\right)$, $D\left(\sqrt{\frac{1}{2}},-\sqrt{\frac{1}{2}}\right)=12-4>0 \Rightarrow$ local min.

$$f_{xx}=2$$